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Multidimensional maps of QRT type

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Abstract
We consider a class of multidimensional maps which naturally generalize the QRT map of the plane. Our $2n$-dimensional maps are volume preserving and have n rational invariants, but we do not generally have a symplectic form. However, many specializations and reductions are integrable, some of which we present. Included in these are some new four-dimensional generalizations of the McMillan map.

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1. Introduction

The largest family of integrable mappings in the plane, known to date, was constructed by Quispel, Roberts and Thompson [9]. These mappings are rational and depend upon 18 parameters, possess an invariant which is fractionally bi-quadratic and are measure preserving (and therefore symplectic). The QRT family can be 'integrated' in terms of elliptic functions.

There are algebro-geometric arguments [12] which suggest that the QRT map is *the most general* bi-rational map of the plane which can be parameterized in terms of elliptic functions, but there are still attempts to find integrable mappings of the plane which *do not* fall into the QRT class [13].

On the other hand, there is no classification of *higher dimensional*, integrable mappings, but a number of interesting examples have been published in recent years. In [3], Capel and Sahadevan constructed a four-dimensional symplectic map with a rational four-quadratic integral. The symplectic form is non-constant and in general their map has only one integral. However, by further reduction, they were able to relate their maps to lattice versions of the MKdV and sine-Gordon equations, with a Lax representation, and thus find a second integral, which was in involution with the first. The same approach was later used by Iatrou [5], who obtained a different four-dimensional map, with a constant symplectic form, together with various higher dimensional ones, also with Lax representations.

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In this paper, we present some *different* higher dimensional generalizations of the QRT maps. Our starting point is a number of vector equations analogous to those introduced by QRT (see (2)). These give rise to pairs of non-commuting involutions, which may be composed to form $2n$-dimensional, multi-parameter maps with n or more *invariants* (first integrals) and, furthermore, are measure preserving. Without further structure it is not possible to say anything about the integrability of our maps. On the intersection of level surfaces of the n first integrals, each map reduces to an n-dimensional, reduced map, but generically has no further structure. It should be pointed out that it may not be possible to write the n-dimensional maps *explicitly* (since this involves solving a collection of polynomial equations in several variables) and that, even when this can be done, the resulting map would not generally be *rational*, but *algebraic*.

By imposing some symmetry constraints on our choice of vectors (generalizing $\mathbf{X}$ and $\mathbf{Y}$ of the QRT construction) and on the choice of parameter matrices (generalizing A_0 and A_1 of the QRT construction), we derive integrable subclasses of our general maps. In particular, we derive some new 4D generalizations of the McMillan map (see section 3.1). In other cases (see section 3.2) an additional first integral leads to a degenerate Poisson bracket, which can then be reduced to lower dimensions and hence to integrability.

Before introducing our maps, we first discuss some of the additional algebraic properties which can be used to deduce integrability.

The Liouville theorem. Veselov [12] extended the usual Liouville theorem to the context of maps. A $2n$-dimensional *symplectic* map with n (functionally independent) *involutive* first integrals is integrable, reducing to a simple shift on the torus (intersection of level surfaces of first integrals) when written in action-angle variables. The n commuting *continuous* flows on the torus are continuous symmetries of the map.

Commuting maps. Veselov also discusses another definition of integrability in [12]. In analogy to the 'symmetry approach' in the theory of integrable nonlinear evolution equations (see [7] and references therein), Veselov defines a map to be integrable if it *commutes* (under the operation of function composition) with another, requiring that their orbits are disjoint (discounting, for example, the commutativity of a map and its double iteration). This was inspired by the work of Ritt [10].

Additional first integrals. With additional integrals, the phase space reduces to less than n dimensions. In the case of measure preserving maps with $2(n-1)$ integrals, we can build a degenerate Poisson bracket with $2(n-1)$ Casimir functions, reducing the map to being two dimensional and symplectic. With $2n-1$ integrals, this two-dimensional symplectic map is integrable.

Decoupled or triangular systems. A map may decouple into integrable sub-maps in a variety of ways. A direct sum of integrable maps (acting on the Cartesian product of phase spaces) is trivially integrable. Some of our examples are *triangular*, meaning that $2k$ variables satisfy a self-contained integrable map with these $2k$ variables acting as parameters in the map satisfied by the remaining $2(n-k)$ variables (see example 3.2 and equation (24)). The integrability of the whole map rests upon the integrability of the second part. In our examples this integrability holds.

Integrability up to quadrature? It should be noted that *none* of these definitions of integrability entail such notions as 'integrability up to quadrature' that we have for ordinary differential equations.

2. The general construction

In this paper we present some generalizations of the QRT map, giving a construction of a family of $2n$-dimensional maps, possessing n independent first integrals. Integrable subfamilies are presented in later sections. The general construction directly mirrors that of QRT in two dimensions, so we first describe this case.

2.1. The QRT map of the plane

Consider the vectors

$$\mathbf{X} = \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}, \qquad \mathbf{Y} = \begin{pmatrix} y^2 \\ y \\ 1 \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} f_1(y) \\ f_2(y) \\ f_3(y) \end{pmatrix}, \qquad \mathbf{g} = \begin{pmatrix} g_1(\tilde{x}) \\ g_2(\tilde{x}) \\ g_3(\tilde{x}) \end{pmatrix}, \tag{1}$$

together with the equations

$$\mathbf{X} \cdot (\mathbf{f} \times \tilde{\mathbf{X}}) = 0, \qquad \mathbf{Y} \cdot (\mathbf{g} \times \tilde{\mathbf{Y}}) = 0, \tag{2}$$

where the 'tilde' refers to vectors of the same form, but written in terms of the variables $\tilde{x}$ and $\tilde{y}$. These equations are *quadratic* in $x, \tilde{x}$ (respectively $y, \tilde{y}$), but possess factors $(\tilde{x} - x)$ (respectively $(\tilde{y} - y)$), so can be *solved* for $\tilde{x}, \tilde{y}$ in *rational form*:

$$\tilde{x} = \frac{f_1(y) - x f_2(y)}{f_2(y) - x f_3(y)}, \qquad \tilde{y} = \frac{g_1(\tilde{x}) - y g_2(\tilde{x})}{g_2(\tilde{x}) - y g_3(\tilde{x})}. \tag{3}$$

The vectors $\mathbf{f}$ and $\mathbf{g}$ have special form

$$\mathbf{f} = (A_0 \mathbf{Y}) \times (A_1 \mathbf{Y}), \qquad \mathbf{g} = \left(A_0^T \tilde{\mathbf{X}}\right) \times \left(A_1^T \tilde{\mathbf{X}}\right),$$

which, when inserted into (2), along with the use of a standard vector identity, leads to

$$\frac{\mathbf{X}^T A_0 \mathbf{Y}}{\mathbf{X}^T A_1 \mathbf{Y}} = \frac{\tilde{\mathbf{X}}^T A_0 \mathbf{Y}}{\tilde{\mathbf{X}}^T A_1 \mathbf{Y}} \qquad \text{and} \qquad \frac{\tilde{\mathbf{X}}^T A_0 \mathbf{Y}}{\tilde{\mathbf{X}}^T A_1 \mathbf{Y}} = \frac{\tilde{\mathbf{X}}^T A_0 \tilde{\mathbf{Y}}}{\tilde{\mathbf{X}}^T A_1 \tilde{\mathbf{Y}}},$$

from which the following *integral* is deduced:

$$I(x, y) = \frac{\mathbf{X}^T A_0 \mathbf{Y}}{\mathbf{X}^T A_1 \mathbf{Y}}, \qquad \text{with} \quad I(\tilde{x}, \tilde{y}) = I(x, y). \tag{4}$$

The 18 parameters of the map and the integral are just the matrix coefficients of A_0 and A_1. An important property of the QRT map is that it is *reversible*, meaning that it can be written as the composition $i_2 \circ i_1$ of two *involutions*:

$$\iota_1 : \begin{cases} \tilde{x} = \dfrac{f_1(y) - x f_2(y)}{f_2(y) - x f_3(y)}, \\ \tilde{y} = y, \end{cases} \qquad \iota_2 : \begin{cases} \tilde{x} = x, \\ \tilde{y} = \dfrac{g_1(\tilde{x}) - y g_2(\tilde{x})}{g_2(\tilde{x}) - y g_3(\tilde{x})}, \end{cases}$$

with $\iota_1^2 = \iota_2^2 = \mathrm{id}$.

2.2. A degenerate case

When the parameter matrices A_i take the special form

$$A_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \varepsilon_i & \xi_i \\ 0 & \lambda_i & \mu_i \end{pmatrix},$$

we have $f_2 = f_3 = g_2 = g_3 = 0$, so the QRT map (3) is ill defined. However, the integral (4) *is* still defined and takes the form

$$I(x,y) = \frac{\varepsilon_0 xy + \lambda_0 y + \xi_0 x + \mu_0}{\varepsilon_1 xy + \lambda_1 y + \xi_1 x + \mu_1}. \tag{5}$$

By considering the two equations $I(\tilde{x}, y) = -I(x, y)$ and $I(x, \tilde{y}) = -I(x, y)$, we obtain the two involutions:

$$\iota_1 : \begin{cases} \tilde{x} = -\dfrac{c_1(b_0x + c_0) + c_0(b_1x + c_1)}{b_1(b_0x + c_0) + b_0(b_1x + c_1)}, \\ \tilde{y} = y, \end{cases} \qquad \iota_2 : \begin{cases} \tilde{x} = x, \\ \tilde{y} = -\dfrac{f_1(e_0y + f_0) + f_0(e_1y + f_1)}{e_1(e_0y + f_0) + e_0(e_1y + f_1)}, \end{cases}$$

where

$$\begin{pmatrix} b_i \\ c_i \end{pmatrix} = \begin{pmatrix} \varepsilon_i & \xi_i \\ \lambda_i & \mu_i \end{pmatrix} \begin{pmatrix} y \\ 1 \end{pmatrix} = A_i \mathbf{Y}, \qquad \begin{pmatrix} e_i \\ f_i \end{pmatrix} = \begin{pmatrix} \varepsilon_i & \lambda_i \\ \xi_i & \mu_i \end{pmatrix} \begin{pmatrix} \tilde{x} \\ 1 \end{pmatrix} = A_i^T \tilde{\mathbf{X}}.$$

With these definitions, $I(x, y)$ can be written as

$$I = \frac{b_0x + c_0}{b_1x + c_1} \qquad \text{or} \qquad I = \frac{e_0y + f_0}{e_1y + f_1},$$

depending upon which involution we are dealing with. The involutions can then be written as

$$\iota_1 : \begin{cases} \tilde{x} = -\dfrac{c_1K + c_0}{b_1K + b_0} = \dfrac{\alpha_1 y + \beta_1}{\gamma_1 y + \delta_1}, \\ \tilde{y} = y, \end{cases} \qquad \iota_2 : \begin{cases} \tilde{x} = x, \\ \tilde{y} = -\dfrac{f_1K + f_0}{e_1K + e_0} = \dfrac{\alpha_2 x + \beta_2}{\gamma_2 x + \delta_2}, \end{cases}$$

where K is the numerical value of the integral $I(x, y)$ for a given orbit and α_i, etc, are just combinations of c_i, etc, and K. By solving $I(x, y) = K$ for y (in the case of ι_1) or for x (in the case of ι_2), each of these involutions reduces to a *fractionally linear* map of one variable and is thus exactly solvable.

2.3. The 2n-dimensional generalization

We build mappings of a $2n$-dimensional space, with coordinates $x_i, y_i, i = 1, \ldots, n$. Consider the set of $2N$ vectors, with $N = \frac{1}{2}n^2(n + 1)$,

$$\mathbf{X}_k^{ij} = \begin{pmatrix} x_i x_j \\ x_k \\ 1 \end{pmatrix}, \qquad \mathbf{Y}_k^{ij} = \begin{pmatrix} y_i y_j \\ y_k \\ 1 \end{pmatrix}, \qquad i, j, k = 1, \ldots, n. \tag{6}$$

From these we choose $2n$ vectors $\mathbf{X}_i, \mathbf{Y}_i, i = 1, \ldots, n$, with which we form the equations

$$\mathbf{X}_i \cdot (\mathbf{f}_i \times \tilde{\mathbf{X}}_i) = 0, \qquad \mathbf{Y}_i \cdot (\mathbf{g}_i \times \tilde{\mathbf{Y}}_i) = 0, \tag{7}$$

where the vectors $\mathbf{f}_i, \mathbf{g}_i$ are defined by

$$\mathbf{f}_i = \begin{pmatrix} f_{3i-2} \\ f_{3i-1} \\ f_{3i} \end{pmatrix} = (A_{i0}\mathbf{Y}_i) \times (A_{i1}\mathbf{Y}_i), \qquad \mathbf{g}_i = \begin{pmatrix} g_{3i-2} \\ g_{3i-1} \\ g_{3i} \end{pmatrix} = \left(A_{i0}^T\tilde{\mathbf{X}}_i\right) \times \left(A_{i1}^T\tilde{\mathbf{X}}_i\right). \tag{8}$$

As in the case of the QRT map, the matrices A_{i0}, A_{i1} are the source of the $18n$ parameters in the map. The maps obtained in this way have n integrals, given by

$$I_k(\mathbf{x}, \mathbf{y}) = \frac{\mathbf{X}_k^T A_{k0} \mathbf{Y}_k}{\mathbf{X}_k^T A_{k1} \mathbf{Y}_k}, \qquad \text{with} \qquad I_k(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = I_k(\mathbf{x}, \mathbf{y}), \qquad k = 1, \ldots, n. \tag{9}$$

There are ${}_N C_n$ choices of the vectors $\mathbf{X}_k^{ij}$ (and a symmetric choice of $\mathbf{Y}_k^{ij}$), each giving either a map or a 2–1 correspondence, but they fall into equivalence classes under the action of the permutation group S_n of n symbols. Generally, we discard the correspondences, concentrating on the maps, which are *bi-rational*. Furthermore, for each n, only some of these maps are genuinely new, since many of them decouple into lower dimensional maps or into *triangular* form. Nevertheless, this construction gives rise to an interesting array of *new* maps of $2n$-dimensional space with n or more integrals. Many subcases are integrable and we have not yet detected any non-integrability.

As with the QRT construction, we generally construct pairs of *non-commuting involutions*, which are composed to form nontrivial infinite order maps, which are therefore *reversible*. For special choices of parameters, the maps can be of *finite* order, in parallel with the usual QRT map (see [11]).

3. Four-dimensional maps

Here, with $n = 2$, we have $N = 6$, giving us 15 possibilities, which fall into nine equivalence classes under the permutation group S_2. The choices $\{\mathbf{X}_1^{11}, \mathbf{X}_2^{22}\}$, $\{\mathbf{X}_2^{11}, \mathbf{X}_1^{22}\}$, $\{\mathbf{X}_1^{12}, \mathbf{X}_2^{12}\}$ are symmetric under S_2. The first is just a pair of uncoupled QRT maps and the second a 2–1 correspondence. The third case is our example 3.1. Of the remaining 12 choices, 6 are 2–1 correspondences and 6 bi-rational, one of which is given in example 3.2.

Example 3.1 (with vectors $\mathbf{X}_1^{12}$ and $\mathbf{X}_2^{12}$)**.** With this choice, the 36-parameter family of maps takes the form

$$\begin{aligned} \tilde{x}_1 &= \frac{(f_1 - x_2 f_2)(f_4 - x_1 x_2 f_6)}{(f_5 - x_2 f_6)(f_1 - x_1 x_2 f_3)}, & \tilde{x}_2 &= \frac{(f_4 - x_1 f_5)(f_1 - x_1 x_2 f_3)}{(f_2 - x_1 f_3)(f_4 - x_1 x_2 f_6)}, \\ \tilde{y}_1 &= \frac{(g_1 - y_2 g_2)(g_4 - y_1 y_2 g_6)}{(g_5 - y_2 g_6)(g_1 - y_1 y_2 g_3)}, & \tilde{y}_2 &= \frac{(g_4 - y_1 g_5)(g_1 - y_1 y_2 g_3)}{(g_2 - y_1 g_3)(g_4 - y_1 y_2 g_6)}, \end{aligned} \tag{10}$$

where $f_i(\mathbf{y})$ and $g_i(\tilde{\mathbf{x}})$ are given by (8). This map has the two integrals I_1, I_2, given by (9), is measure preserving with the density

$$m(x_1, x_2, y_1, y_2) = \frac{1}{\left(\mathbf{X}_1^T A_{10} \mathbf{Y}_1\right)\left(\mathbf{X}_2^T A_{21} \mathbf{Y}_2\right)}, \tag{11}$$

and is reversible, since it can be written in the form $\iota_2 \circ \iota_1$, where the involutions ι_k are given by

$$\iota_1 : \begin{cases} \tilde{x}_1 = \dfrac{(f_1 - x_2 f_2)(f_4 - x_1 x_2 f_6)}{(f_5 - x_2 f_6)(f_1 - x_1 x_2 f_3)}, & \tilde{x}_2 = \dfrac{(f_4 - x_1 f_5)(f_1 - x_1 x_2 f_3)}{(f_2 - x_1 f_3)(f_4 - x_1 x_2 f_6)}, \\ \tilde{y}_1 = y_1, & \tilde{y}_2 = y_2, \end{cases} \tag{12}$$

$$\iota_2 : \begin{cases} \tilde{x}_1 = x_1, & \tilde{x}_2 = x_2, \\ \tilde{y}_1 = \dfrac{(g_1 - y_2 g_2)(g_4 - y_1 y_2 g_6)}{(g_5 - y_2 g_6)(g_1 - y_1 y_2 g_3)}, & \tilde{y}_2 = \dfrac{(g_4 - y_1 g_5)(g_1 - y_1 y_2 g_3)}{(g_2 - y_1 g_3)(g_4 - y_1 y_2 g_6)}. \end{cases} \tag{13}$$

Each of these involutions *anti-preserves* the above measure.

The vectors $\mathbf{X}_i$, $\mathbf{Y}_i$ of this example are symmetric under the permutation $1 \leftrightarrow 2$. For the map (10) to have this symmetry, we must restrict the vectors $\mathbf{f}_i$ so that $\mathbf{f}_1 = \mathbf{f}_2$, which means that the matrices A_{ki} must satisfy $A_{10} = A_{20}$ and $A_{11} = A_{21}$. Under these restrictions, the map is invariant and the integrals interchange under the above permutation, which corresponds to a simple evolution of phase space, given by

$$\hat{x}_1 = x_2, \qquad \hat{x}_2 = x_1, \qquad \hat{y}_1 = y_2, \qquad \hat{y}_2 = y_1. \tag{14}$$

The maps (10) and (14) *commute* and $I_1 \leftrightarrow I_2$ under the latter.

This symmetrically coupled map reduces to the QRT when $(x_2, y_2) \to (x_1, y_1)$.

Example 3.2 (with vectors $\mathbf{X}_1^{11}$ and $\mathbf{X}_2^{12}$)**.** With this choice, the 36-parameter family of maps takes the form

$$\tilde{x}_1 = \frac{(f_1 - x_1 f_2)}{(f_2 - x_1 f_3)}, \qquad \tilde{x}_2 = \frac{(f_4 - x_1 f_5)x_2}{f_4 - \tilde{x}_1 f_5 + (\tilde{x}_1 - x_1)x_2 f_6},$$
$$\tilde{y}_1 = \frac{(g_1 - y_1 g_2)}{(g_2 - y_1 g_3)}, \qquad \tilde{y}_2 = \frac{(g_4 - y_1 g_5)y_2}{g_4 - \tilde{y}_1 g_5 + (\tilde{y}_1 - y_1)y_2 g_6},$$

where $f_i(\mathbf{y})$ and $g_i(\tilde{\mathbf{x}})$ are given by (8). Once again, this map is measure preserving and reversible. This consists of a self-contained QRT map of the (x_1, y_1) plane, weakly coupled to a rational map of the (x_2, y_2) variables, with coefficients which depend upon (x_1, y_1). The two integrals split in a similar way. The first integral I_1 is just the original QRT integral, as follows from the choice of $\mathbf{X}_1, \mathbf{Y}_1$. On the other hand, the second integral takes the form

$$I_2 = \frac{c_1 x_2 y_2 + c_2 x_2 + c_3 y_2 + c_4}{d_1 x_2 y_2 + d_2 x_2 + d_3 y_2 + d_4},$$

where c_i, d_i the are polynomial functions of the variables x_1, y_1. Using the argument following the integral (5) of the degenerate case of the QRT map, we solve $I_2 = K$ for y_2 to give the latter as a fractionally linear expression in x_2, with coefficients depending upon x_1, y_1. Substituting this into the expression for $\tilde{x}_2$ gives a simple fractionally linear formula:

$$\tilde{x}_2 = \frac{Ax_2 + B}{Cx_2 + D},$$

where, once again, $A, \ldots, D$ are the polynomial expressions in x_1, y_1. In summary, we have that x_2, y_2 evolve by a pair of fractionally linear maps, whose coefficients depend upon the solution of the QRT map, which is itself integrable, so our triangular map is integrable.

3.1. Four-dimensional analogues of the two-dimensional McMillan map

It is also possible to create four-dimensional generalizations of any of the standard reductions of the QRT map. These are produced by special choices of the parameters in the matrices (A_{i0}, A_{i1}). By choosing the matrices as

$$A_{10} = A_{20} = \begin{pmatrix} 1 & 0 & -b \\ 0 & -2\alpha & 0 \\ -b & 0 & b^2 \end{pmatrix}, \qquad A_{11} = A_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we obtain coupled McMillan type maps, the specific form depending upon the choice of vectors $\mathbf{X}_i, \mathbf{Y}_i$.

We would normally have $b = 1$, but in the next example we also wish to consider the reduction with $b = 0$.

Example 3.3 (The symmetrically coupled map)**.** Symmetrically coupled map of example 3.1 reduces to

$$\tilde{x}_1 = y_1, \qquad \tilde{x}_2 = y_2,$$
$$\tilde{y}_1 = -\frac{x_2 y_2}{y_1} - \frac{2\alpha y_2}{b - y_1 y_2}, \qquad \tilde{y}_2 = -\frac{x_1 y_1}{y_2} - \frac{2\alpha y_1}{b - y_1 y_2}, \tag{15}$$

with integrals

$$I_1 = (b - x_1 x_2)(b - y_1 y_2) - 2\alpha x_1 y_1, \qquad I_2 = (b - x_1 x_2)(b - y_1 y_2) - 2\alpha x_2 y_2. \tag{16}$$

This map is *volume* preserving. This is *different* from the four-dimensional generalization which arises through periodic reductions of the lattice KdV equation (see [4]) and is a particular case of the map (10), subject to the restrictions that it commutes with the involution (14).

Defining the first integral $K = \frac{1}{2\alpha}(I_2 - I_1) = x_1y_1 - x_2y_2$, we consider the *reduction* of the map to the three-dimensional manifold defined by the level surface $K = k$. First, we rewrite the four-dimensional map as

$$\begin{aligned} \tilde{x}_1 &= y_1, & \tilde{x}_2 &= y_2, \\ \tilde{y}_1 &= -x_1 + \frac{k}{y_1} - \frac{2\alpha y_2}{b - y_1y_2}, & \tilde{y}_2 &= -x_2 - \frac{k}{y_2} - \frac{2\alpha y_1}{b - y_1y_2}. \end{aligned} \tag{17}$$

Using the first two of these to replace y_i, this map is canonical, with generating function

$$S = x_1\tilde{x}_1 + x_2\tilde{x}_2 + k\log\left(\frac{\tilde{x}_2}{\tilde{x}_1}\right) - 2\alpha\log(b - \tilde{x}_1\tilde{x}_2).$$

Remark 3.1. The map (17), with $\alpha = 1$, $k = 0$, occurs in [1] (equation (3.38)) in the context of a stationary, discrete NLS equation, possessing a Lax pair.

Now use $y_2 = \frac{x_1y_1-k}{x_2}$ to reduce the map to three dimensions

$$\tilde{x}_1 = y_1, \qquad \tilde{x}_2 = \frac{x_1y_1 - k}{x_2}, \qquad \tilde{y}_1 = -x_1 + \frac{k}{y_1} - \frac{2\alpha(x_1y_1 - k)}{bx_2 + ky_1 - x_1y_1^2}, \tag{18}$$

with *degenerate symplectic form*

$$\omega = \begin{pmatrix} 0 & -\frac{y_1}{x_2} & 1 \\ \frac{y_1}{x_2} & 0 & \frac{x_1}{x_2} \\ -1 & -\frac{x_1}{x_2} & 0 \end{pmatrix}.$$

In these coordinates

$$I_1 = \frac{(b - x_1x_2)\left(bx_2 + ky_1 - x_1y_1^2\right)}{x_2} - 2\alpha x_1y_1$$

and $I_2 = I_1 + 2\alpha k$ is redundant. The null vector of ω is $\mathbf{n} = (-x_1, x_2, y_1)$, which satisfies $\mathbf{n}\cdot\nabla I_1 = 0$ and is transformed to $-\mathbf{n}$ under the map (18). This means that the *integral curves* (as unparameterized geometric curves) are invariant under the map, so the three-dimensional space separates. We may adapt coordinates to the vector field (using the *invariants* of $\mathbf{n}$ as two of them) to obtain

$$u = x_1x_2, \qquad v = x_1y_1, \qquad w = y_1, \qquad \text{so} \quad \mathbf{n} = (0, 0, w),$$

and the map becomes

$$\tilde{u} = \frac{v(v-k)}{u}, \qquad \tilde{v} = \frac{(k-v)(bu + (2\alpha + k - v)v)}{bu + (k-v)v}, \qquad \tilde{w} = \frac{\tilde{v}}{w}.$$

We see that the u–v plane is invariant and also that

$$I_1 = \frac{(b-u)(bu + (k-v)v)}{u} - 2\alpha v \qquad \text{and} \qquad \omega = \begin{pmatrix} 0 & \frac{1}{u} & 0 \\ -\frac{1}{u} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This means that the three-dimensional map is *triangular*, with the two-dimensional map of the u–v plane having a first integral and being measure preserving, rendering it completely integrable.

Canonical coordinates are $x = u$, $y = \frac{v}{u}$, in terms of which the two-dimensional map is

$$\tilde{x} = y(xy - k), \qquad \tilde{y} = -\frac{b + (2\alpha + k)y - xy^2}{y(b + ky - xy^2)} \tag{19}$$

and

$$I_1 = (b - x)(b + y(k - xy)) - 2\alpha xy, \qquad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We may construct the particular matrices A_0 and A_1, such that the corresponding QRT map has the integral I_1. We have

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ -b & -2\alpha - k & -b \\ 0 & bk & b^2 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$\begin{aligned} \bar{x} &= \frac{b(1 + y^2) + (2\alpha + k)y - xy^2}{y^2}, \\ \bar{y} &= -\frac{xy^4(xy - 2\alpha - k) + by^2(2\alpha + k - y(2x - ky + xy^2)) + b^2 y(y^2 + 1)}{(b + (2\alpha + k)y - xy^2)(b(1 + y^2) + (2\alpha + k)y - xy^2)}. \end{aligned} \tag{20}$$

The maps (19) and (20) *commute*.

All of our coordinate changes are *bi-rational*, so from the solution of the map (19) we can reverse our steps to obtain the solution of the original four-dimensional McMillan map (15).

The case $b = 0$. We now consider the case of (15) with the condition that $b = 0$:

$$\varphi: \quad \tilde{x}_1 = y_1, \qquad \tilde{x}_2 = y_2, \qquad \tilde{y}_1 = \frac{2\alpha - x_2 y_2}{y_1}, \qquad \tilde{y}_2 = \frac{2\alpha - x_1 y_1}{y_2}, \tag{21}$$

with integrals

$$I_1 = x_1 y_1 (x_2 y_2 - 2\alpha), \qquad I_2 = x_2 y_2 (x_1 y_1 - 2\alpha). \tag{22}$$

Whilst all of our previous formulae hold in this case, we now find that φ preserves the four-dimensional Poisson matrix

$$\mathbf{P} = \begin{pmatrix} \mathbf{0} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{0} \end{pmatrix}, \qquad \text{where} \quad \mathbf{X} = \begin{pmatrix} \frac{2\alpha - x_1 y_1}{2\alpha - x_2 y_2} & \frac{y_2}{y_1} \\ \frac{x_2}{x_1} & 1 \end{pmatrix}. \tag{23}$$

We have

$$\varphi_* \mathbf{P} = \mathbf{P}, \qquad I_k \circ \varphi = I_k, \qquad \{I_1, I_2\} = (\nabla I_1)^T \mathbf{P} \nabla I_2 = 0,$$

so the map φ is Liouville integrable *in the four-dimensional space*.

However, it is also possible to reduce this Poisson bracket to three and then to two dimensions. We again define $K = x_1 y_1 - x_2 y_2$ and write the Poisson matrix $\mathbf{P}_K$ of the coordinates (x_1, y_1, K, x_2):

$$\mathbf{P}_K = \left(\begin{array}{ccc|c} 0 & \frac{2\alpha - x_1 y_1}{2\alpha + K - x_1 y_1} & \frac{K(2\alpha + K - 2x_1 y_1)}{y_1(2\alpha + K - x_1 y_1)} & 0 \\ -\frac{2\alpha - x_1 y_1}{2\alpha + K - x_1 y_1} & 0 & -\frac{K(2\alpha + K - 2x_1 y_1)}{x_1(2\alpha + K - x_1 y_1)} & -\frac{x_2}{x_1} \\ -\frac{K(2\alpha + K - 2x_1 y_1)}{y_1(2\alpha + K - x_1 y_1)} & \frac{K(2\alpha + K - 2x_1 y_1)}{x_1(2\alpha + K - x_1 y_1)} & 0 & 0 \\ \hline 0 & \frac{x_2}{x_1} & 0 & 0 \end{array}\right).$$

This Poisson matrix is invariant under the mapping (when written in these coordinates)

$$\varphi_{k4}(x_1, y_1, K, x_2) = \left(y_1, \frac{2\alpha + K - x_1 y_1}{y_1}, K, \frac{x_1 y_1 - K}{x_2} \right)$$

and the 3×3 block, $\mathbf{P}_3$ is invariant under the three-dimensional reduction $\varphi_{k3}(x_1, y_1, K)$. This map has integrals $I_1 = x_1 y_1 (x_1 y_1 - K - 2\alpha)$ and K (with $I_2 = I_1 + 2\alpha K$), and the Poisson matrix $\mathbf{P}_3$ has Casimir function $\mathcal{C} = \frac{I_1}{K}$. On the symplectic leaves $\mathcal{C} = s$, the map and Poisson matrix reduce to

$$\varphi_2(x_1, y_1) = \left(y_1, \frac{s(2\alpha - x_1 y_1)}{y_1(s + x_1 y_1)} \right), \qquad \mathbf{P}_2 = \begin{pmatrix} 0 & 1 + \frac{x_1 y_1}{s} \\ -1 - \frac{x_1 y_1}{s} & 0 \end{pmatrix}.$$

On this manifold, the integral takes the form

$$I_1 = \frac{s x_1 y_1 (x_1 y_1 - 2\alpha)}{x_1 y_1 + s}.$$

Whilst it is possible to explicitly construct *canonical* coordinates, we have not managed to find ones for which the map and integral take *rational* form.

Example 3.4 (The weakly coupled map)**.** The *weakly coupled* map (3.2) reduces to

$$\begin{aligned} \tilde{x}_1 &= y_1, & \tilde{x}_2 &= y_2, \\ \tilde{y}_1 &= -x_1 - \frac{2\alpha y_1}{1 - y_1^2}, & \tilde{y}_2 &= -\frac{x_2\left(1 - y_1^2\right)(2\alpha y_2 + x_1(1 - y_1 y_2))}{2\alpha(y_1 - y_2) + x_1(1 - y_1^2)(1 - y_1 y_2)}, \end{aligned} \tag{24}$$

with integrals

$$I_1 = \left(1 - x_1^2\right)\left(1 - y_1^2\right) - 2\alpha x_1 y_1, \qquad I_2 = (1 - x_1 x_2)(1 - y_1 y_2) - 2\alpha x_2 y_2, \tag{25}$$

where we have put $b = 1$ in the matrices A_{i0}. This map preserves the measure with density $m = \frac{1}{x_1 x_2}$. This map is integrable in the same way as example (3.2) and is a much simpler calculation, since the formula $\tilde{x}_2 = y_2$ immediately becomes fractionally linear in x_2 upon solving $I_2 = K$ for y_2. This time x_1, y_1 are solutions of the usual McMillan map.

3.2. Four-dimensional maps with three first integrals

Consider again the *symmetrically coupled* map of example 3.1. We now restrict to the 24-parameter subfamily defined by the matrices

$$A_{1i} = \begin{pmatrix} \alpha_{1i} & \beta_{1i} & 0 \\ \delta_{1i} & \epsilon_{1i} & \zeta_{1i} \\ 0 & \zeta_{1i} & \mu_{1i} \end{pmatrix}, \qquad A_{2i} = \begin{pmatrix} \alpha_{2i} & \beta_{2i} & 0 \\ \beta_{2i} & \epsilon_{2i} & \zeta_{2i} \\ 0 & \lambda_{2i} & \mu_{2i} \end{pmatrix}, \qquad i = 0, 1.$$

This choice leads to a symmetry of the integrals I_1, I_2 under the exchange $x_1 \leftrightarrow y_1$: $I_k(x_1, x_2, y_1, y_2) = I_k(y_1, x_2, x_1, y_2)$, which are therefore invariant under the involution

$$\iota_{11} : \hat{x}_1 = y_1, \qquad \hat{x}_2 = x_2, \qquad \hat{y}_1 = x_1, \qquad \hat{y}_2 = y_2. \tag{26}$$

Composing the involution ι_1 of (12) and ι_{11}, we obtain the four-dimensional map $\varphi = \iota_{11} \circ \iota_1$ which possesses three integrals (the third being y_2). In this way the map reduces to three dimensions, possessing two integrals, I_1 and I_2, with y_2 acting as a parameter.

Since the map is measure preserving, we may use one of these integrals to construct a degenerate Poisson bracket (see [2]), which thus reduces to the level surfaces of its Casimir function, which are symplectic leaves. In this way the mapping is two-dimensional, symplectic and possesses *one* first integral (the other having been the Casimir function), so is integrable. In fact, we may use either of the two integrals (or even any function of them) to carry out this construction. Starting with any first integral I and the volume element (with density (11)), we make the following *contraction*:

$$\Omega_I = \left(m \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_1} \right) \lrcorner \, \mathrm{d}I$$
$$= m \left(\frac{\partial I}{\partial y_1} \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} - \frac{\partial I}{\partial x_2} \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \frac{\partial I}{\partial x_1} \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_1} \right).$$

This is degenerate, with I as its Casimir function. Since both the volume element and the function I are invariant under the action of the map, so is Ω. A second invariant function then generates a continuous symmetry of the map. In particular, we write Ω_j to represent the Poisson bi-vector (also the corresponding matrix) with Casimir I_j and write the continuous symmetry in bi-Hamiltonian form:

$$0 = \Omega_1 \nabla I_1, \qquad \Omega_1 \nabla I_2 = -\Omega_2 \nabla I_1, \qquad \Omega_2 \nabla I_2 = 0. \tag{27}$$

When reduced to the two-dimensional symplectic leaves, the map reduces to the standard QRT map of the plane.

The *explicit* form of these calculations is complicated for the general set of parameters, so we consider a simple example with *polynomial* first integrals by choosing

$$A_{i1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

With the given form of the matrices A_{i0}, the integrals are

$$I_1 = \mu_{10} + \zeta_{10}(x_1 + y_1) + x_1 y_1 (\alpha_{10} x_2 y_2 + \beta_{10} x_2 + \delta_{10} y_2 + \epsilon_{10}),$$
$$I_2 = \mu_{20} + \zeta_{20} x_2 + \lambda_{20} y_2 + x_2 y_2 (\alpha_{20} x_1 y_1 + \beta_{20}(x_1 + y_1) + \epsilon_{10}).$$

The three-dimensional Poisson matrix Ω_1, written in terms of the coordinates x_1, y_1, I_1, is

$$\Omega_1 = m \begin{pmatrix} 0 & -x_1 y_1 (\beta_{10} + \alpha_{10} y_2) & 0 \\ x_1 y_1 (\beta_{10} + \alpha_{10} y_2) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

With formula (11), we see that in this case $m = I_1$, so we may omit it.

Elimination of x_2. On the level surface $I_1 = s$ we can eliminate x_2 to obtain a rational map in x_1, y_1, with *parameters* y_2, s, as well as those of the matrices A_{i0}, and the integral I_2 takes *rational* form. We may build the QRT map of the x_1–y_1 plane with this integral and this turns out to be the *double iteration* of our map. This is best illustrated by choosing very specific examples of the matrices A_{i0}, such as

$$A_{10} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A_{20} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

The map is then

$$\varphi(x_1, y_1) = \left(y_1, \frac{(s - y_1)(y_1 y_2 + 1)}{x_1 y_2 (y_1 y_2 - 1)} \right),$$

and the first integral

$$I_2 = \frac{(s - x_1 - y_1)(1 + (x_1 + y_1) y_2)}{x_1 y_1} + (x_1 + y_1) y_2^2.$$

This first integral corresponds to a QRT map of the plane with matrices

$$A_0 = \begin{pmatrix} 0 & y_2^2 & -y_2 \\ y_2^2 & -2y_2 & sy_2 - 1 \\ -y_2 & sy_2 - 1 & s \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The corresponding QRT map is the composition $j_2 \circ j_1$ of the two involutions:

$$j_1(x_1, y_1) = \left(\frac{(s - y_1)(y_1 y_2 + 1)}{x_1 y_2 (y_1 y_2 - 1)}, y_1 \right), \qquad j_2(x_1, y_1) = \left(x_1, \frac{(s - x_1)(x_1 y_2 + 1)}{y_1 y_2 (x_1 y_2 - 1)} \right).$$

It can be checked directly that $j_2 \circ j_1 = \varphi \circ \varphi$, but that φ does not commute with either of these involutions.

Elimination of y_1. On the level surface $I_1 = s$ we may, as an alternative, eliminate y_1 to obtain a rational map in x_1, x_2, with *parameters* y_2, s, as well as those of the matrices A_{i0}, and the integral I_2 again takes *rational* form. With the same choice of A_{10}, A_{20} as above, we obtain

$$\varphi(x_1, x_2) = \left(\frac{s - x_1}{1 + x_1(x_2 - y_2)}, \frac{(sy_2 - 1 - x_1 x_2)(1 + sy_2 + x_1(1 + x_1 y_2)(x_2 - y_2))}{(s - x_1)(1 + sy_2 + x_1(x_2 - 2y_2))} \right),$$

$$I_2 = \frac{x_2(1 + x_1 x_2) + \left(x_1^2 x_2^2 - 2x_1 x_2 + sx_2 - 1\right) y_2 + x_1(1 - x_1 x_2) y_2^2}{1 + x_1(x_2 - y_2)}.$$

We may calculate the matrices A_0 and A_1 for which this I_2 is a QRT integral in the x_1–x_2 plane. It turns out that the QRT map is *exactly* our map $\varphi(x_1, x_2)$.

4. Higher dimensional maps

Whilst the general construction of section 2 can be carried out in any dimension, the explicit formulae become complicated for general parameters, so it is more profitable to choose interesting reductions, such as the *coupled McMillan maps* of section 3.1 or that of section 3.2. An additional problem is that of finding an appropriate symplectic form, since we have no general construction for this.

To illustrate this without filling the paper with too many unwieldy formulae we just present one simple example, and even here specify simple numerical values for all the (in this case) 63 parameters.

Example 4.1 (six-dimensional maps with five integrals)**.** Consider the choice of vectors $\mathbf{X}_1^{12}$, $\mathbf{X}_2^{23}$ and $\mathbf{X}_3^{31}$, giving rise to the map

$$\iota_1 = \begin{cases} \tilde{x}_1 = \dfrac{(f_1 - x_2 f_2)\left(x_1 x_2 x_3^2 f_6 f_9 - x_1 x_2 x_3 f_6 f_8 - x_1 x_3 (f_4 f_9 - f_5 f_8) - x_3 f_5 f_7 + f_4 f_7\right)}{(f_8 - x_3 f_9)\left(x_1 x_2^2 x_3 f_3 f_6 - x_1 x_2 x_3 f_3 f_5 - x_2 x_3 (f_1 f_6 - f_2 f_5) - x_2 f_2 f_4 + f_1 f_4\right)}, \\ \tilde{x}_2 = \dfrac{(f_4 - x_3 f_5)\left(x_1^2 x_2 x_3 f_3 f_9 - x_1 x_2 x_3 f_2 f_9 - x_1 x_2 (f_2 f_8 - f_3 f_7) - x_1 f_1 f_8 + f_1 f_7\right)}{(f_2 - x_1 f_3)\left(x_1 x_2 x_3^2 f_9 f_6 - x_1 x_2 x_3 f_6 f_8 - x_1 x_3 (f_4 f_9 - f_5 f_8) - x_3 f_5 f_7 + f_4 f_7\right)}, \\ \tilde{x}_3 = \dfrac{(f_7 - x_1 f_8)\left(x_1 x_2^2 x_3 f_3 f_6 - x_1 x_2 x_3 f_3 f_5 - x_2 x_3 (f_1 f_6 - f_2 f_5) - x_2 f_2 f_4 + f_1 f_4\right)}{(f_5 - x_2 f_6)\left(x_1^2 x_2 x_3 f_3 f_9 - x_1 x_2 x_3 f_2 f_9 - x_1 x_2 (f_3 f_7 - f_2 f_8) - x_1 f_1 f_8 + f_1 f_7\right)}, \\ \tilde{y}_1 = y_1, \\ \tilde{y}_2 = y_2, \\ \tilde{y}_3 = y_3. \end{cases}$$

The 63-parameter involution corresponding to the choice of matrices

$$A_{1i} = \begin{pmatrix} \alpha_{1i} & \beta_{1i} & 0 \\ \delta_{1i} & \epsilon_{1i} & \zeta_{1i} \\ 0 & \zeta_{1i} & \mu_{1i} \end{pmatrix}, \qquad A_{2i} = \begin{pmatrix} \alpha_{2i} & \beta_{2i} & \gamma_{2i} \\ \delta_{2i} & \epsilon_{2i} & \zeta_{2i} \\ \kappa_{2i} & \lambda_{2i} & \mu_{2i} \end{pmatrix},$$

$$A_{3i} = \begin{pmatrix} \alpha_{3i} & \beta_{3i} & 0 \\ \beta_{3i} & \epsilon_{3i} & \zeta_{3i} \\ 0 & \lambda_{3i} & \mu_{3i} \end{pmatrix}, \qquad i = 0, 1$$

has integrals I_1, I_2 and I_3 of the form (9), which possess the discrete symmetry $1 \leftrightarrow 2$, making them invariant under the second involution

$$\iota_{11} : \tilde{x}_1 = y_1 \qquad \tilde{x}_2 = x_2 \qquad \tilde{x}_3 = x_3 \qquad \tilde{y}_1 = x_1 \qquad \tilde{y}_2 = y_2 \qquad \tilde{y}_3 = y_3.$$

Composing ι_1 and ι_{11} we obtain a six-dimensional mapping $\phi = \iota_{11} \circ \iota_1$ with five integrals (the fourth and fifth being y_2 and y_3). By considering y_2, y_3 as *parameters*, we have a four-dimensional measure preserving mapping with three integrals. The formula for the measure is

$$m(x_1, x_2, x_3, y_1) = \frac{1}{\left(\mathbf{X}_1^T A_{10} \mathbf{Y}_1\right)\left(\mathbf{X}_2^T A_{21} \mathbf{Y}_2\right)\left(\mathbf{X}_3^T A_{30} \mathbf{Y}_3\right)}. \tag{28}$$

Following the approach of section 3.2, we build three degenerate Poisson tensors. Choosing any two first integrals I, J, we have

$$\Omega(I, J) = \left(m \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial y_1} \right) \lrcorner \mathrm{d}I \lrcorner \mathrm{d}J,$$

giving

$$\begin{aligned} \frac{1}{m}\Omega(I, J) = & \left(\frac{\partial I}{\partial y_1}\frac{\partial J}{\partial x_3} - \frac{\partial I}{\partial x_3}\frac{\partial J}{\partial y_1} \right) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \left(\frac{\partial I}{\partial x_2}\frac{\partial J}{\partial y_1} - \frac{\partial I}{\partial y_1}\frac{\partial J}{\partial x_2} \right) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} \\ & + \left(\frac{\partial I}{\partial x_3}\frac{\partial J}{\partial x_2} - \frac{\partial I}{\partial x_2}\frac{\partial J}{\partial x_3} \right) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \left(\frac{\partial I}{\partial y_1}\frac{\partial J}{\partial x_1} - \frac{\partial I}{\partial x_1}\frac{\partial J}{\partial y_1} \right) \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \\ & + \left(\frac{\partial I}{\partial x_1}\frac{\partial J}{\partial x_3} - \frac{\partial I}{\partial x_3}\frac{\partial J}{\partial x_1} \right) \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_1} + \left(\frac{\partial I}{\partial x_2}\frac{\partial J}{\partial x_1} - \frac{\partial I}{\partial x_1}\frac{\partial J}{\partial x_2} \right) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial y_1}. \end{aligned}$$

This has rank 2, with two Casimir functions, I and J. With (i, j, k) being a cyclic permutation of $(1, 2, 3)$, we define the three degenerate Poisson tensors

$$\Omega^i = \Omega(I_j, I_k).$$

Using the same symbol to denote the Poisson matrix, we have the following relations:

$$\Omega^i \nabla I_j = 0, \qquad i \neq j \qquad \text{and} \qquad \Omega^1 \nabla I_1 = \Omega^2 \nabla I_2 = \Omega^3 \nabla I_3. \tag{29}$$

Since the integrals and the volume element are invariant under the action of the mapping, so are the three Poisson matrices. The tri-Hamiltonian flow of (29) is a continuous symmetry of the map and the trajectory is just the intersection of the five integrals in the original six-dimensional space. Since each Poisson matrix has rank 2, the four-dimensional manifold defined by (x_1, x_2, x_3, y_1) can be foliated by the two-dimensional symplectic leaves of the Poisson matrix, these being the level sets of the pair of Casimir functions.

To illustrate this construction, we consider a special case of the parameter matrices:

$$A_{10} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A_{20} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_{30} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad A_{i1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for $i = 1, 2, 3$, leading to the integrals

$$I_1 = x_1 + y_1 + x_1y_1(x_2 - y_2), \qquad I_2 = 1 + x_2y_2(1 + x_3y_3),$$
$$I_3 = x_3 - y_3 + x_3y_3(x_1 + y_1).$$

We only give the explicit expression for one of the Poisson tensors:

$$\begin{aligned}\Omega^2 = {} & (1 + x_1x_2 - x_1y_2)(1 + x_1y_3 + y_1y_3)\partial_{x_1} \wedge \partial_{x_2} + x_1x_3y_1y_3\partial_{x_1} \wedge \partial_{x_3} \\ & - x_1y_1(1 + x_1y_3 + y_1y_3)\partial_{x_1} \wedge \partial_{y_1} + x_3y_3(x_1 - y_1)(x_2 - y_2)\partial_{x_2} \wedge \partial_{x_3} \\ & + (1 + x_2y_1 - y_1y_2)(1 + x_1y_3 + y_1y_3)\partial_{x_2} \wedge \partial_{y_1} + x_1x_3y_1y_3\partial_{x_3} \wedge \partial_{y_1}.\end{aligned}$$

On the level surfaces of its Casimir functions, $I_1 = r$ and $I_3 = s$, we may eliminate x_2, x_3 to obtain a rational map $\varphi_2(x_1, y_1)$ of the plane:

$$\tilde{x}_1 = y_1,$$
$$\tilde{y}_1 = \frac{(r - y_1)(1 + x_1y_3 + y_1y_3)(1 + y_3(s + y_1 + y_3))}{y_3(y_1 - r + y_3((s - r + y_1)(x_1 + y_1) - sx_1y_1y_2 - rs) + y_3^2(x_1 + y_1 - x_1y_1y_2 - r))}.$$

With respect to these coordinates, the integral I_2 (up to an additive constant) takes the following rational form:

$$I_2(x_1, y_1) = \frac{y_2(r - x_1 - y_1 + x_1y_1y_2)(1 + y_3(s + x_1 + y_1 + y_3))}{x_1y_1(1 + x_1y_3 + y_1y_3)}. \qquad (30)$$

To reduce the Poisson matrix of Ω^2 to this manifold, we write it in terms of the coordinates: (x_1, y_1, r, s)

$$\Omega^2 = \left(\begin{array}{cc|cc} 0 & x_1y_1(1 + (x_1 + y_1)y_3) & 0 & 0 \\ -x_1y_1(1 + (x_1 + y_1)y_3) & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Once again, we can calculate the QRT map of the plane which corresponds to the integral (30). Once again this is just the double iteration of $\varphi_2(x_1, y_1)$, even though this map does not commute with either of the constituent involutions.

Further discussion and examples of higher dimensional maps can be found in [6].

5. Conclusions

In this paper, we have presented an interesting class of maps of $2n$-dimensional space, possessing n first integrals. When $n = 1$, this is just the QRT map and the construction for general n is a direct generalization of the QRT construction. We have no *general* proof of the integrability of our maps, but have presented several integrable subclasses. Currently, we have no general technique of constructing an invariant symplectic form, except in the *super-integrable* case, with $2n - 1$ invariant functions. However, we believe that there are many integrable cases hidden within this family. Furthermore, to each of our maps it is possible to follow the procedure of Quispel [8] to obtain a corresponding *alternating map*, whose square is our generalized QRT map.

A natural question is how to isolate and classify integrable cases of our maps and this is the most important task for the future. An efficient proof of integrability would be the construction of a symplectic structure, which is also an important task for the future.

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